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Criteria for the asymptotic behaviour of solutions to certain third-order nonlinear differential equations

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Abstract. We investigate and provide sufficient criteria for ultimate boundedness of solutions as well as the asymptotic stability of the trivial solution to certain nonlinear differential equation of order three. Using the Lyapunov second method and the Yoshizawa limit point approach, we establish our results. The equation considered is new and more general. Hence, our results are new, generalized and improved on some earlier established results. In order to verify the correctness of our obtained results, a numerical example is provided with the trajectories of the solutions.

1. Introduction

The discovery of differential equation(DE) between 16th and 17th centuries as a branch of mathematics has paved way for modeling of real life problems of interest in science, technology, finance and many other areas of human life, see [10, 12, 15, 19, 24] and the references contained in them. Thus, there has been a continuous need to examine the qualitative properties of solutions of various DEs in literature. In most of the papers found on the qualitative study of solution of DEs in literature, the direct method of Lyapunov has been considered a veritable and celebrated method being used by several notable researchers. This method, needs no knowledge of the solutions of the equation being investigated before studying the qualitative behavour of the solutions of any DE. The method requires constructing a function that is positive definite everywhere in the domain of definition,

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except at the equilibrium where it becomes zero, while the derivative of the function along the solution curves to the DE under study is expected to be negative semi-definite. However, to construct such a function is and remains a tedious task for researchers, especially if the equation is non-linear.

Despite the above mentioned challenge of the method, it has been extensively employed to establish many interesting results in the area of qualitative behaviour of solutions of many classess of DEs. See for instance references [1–3, 6–9, 11, 13–18, 20, 21, 23, 24, 26] are few among several authors who have employed the method.

Hence, this paper is about

$$\begin{aligned}&\dddot{\varphi} + (u(t,\varphi,\dot{\varphi},\ddot{\varphi}) + v(t,\varphi,\dot{\varphi},\ddot{\varphi}))\ddot{\varphi}\\ &\quad + w(t,\varphi,\dot{\varphi},\ddot{\varphi})\dot{\varphi} + r(t)c(t,\varphi,\dot{\varphi},\ddot{\varphi})h(\varphi) = p(t,\varphi,\dot{\varphi},\ddot{\varphi}),\end{aligned} \tag{1}$$

where $h \in C(\mathbb{R},\mathbb{R}), r(t) \in C(\mathbb{R}^+,\mathbb{R})$ $u, v, w, c, p \in C\big([0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R},\, \mathbb{R}\big)$, $\mathbb{R} = (-\infty,\infty)$ and $\mathbb{R}^+ = [0,+\infty)$. Equation (1) is reduced to system of first order DEs:

$$\begin{aligned}\dot{\varphi} &= \chi,\\ \dot{\chi} &= \psi,\\ \dot{\psi} &= -(u(t,\varphi,\chi,\psi) + v(t,\varphi,\chi,\psi))\psi - w(t,\varphi,\chi,\psi)\chi\\ &\qquad - r(t)c(t,\varphi,\chi,\psi)h(\varphi) + p(t,\varphi,\chi,\psi).\end{aligned} \tag{2}$$

It shall be taken that the partial derivatives

$$w_t(t,\varphi,\chi,\psi), \quad w_\varphi(t,\varphi,\chi,\psi), \quad w_\psi(t,\varphi,\chi,\psi), \quad \dot{r}(t) = \frac{dr(t)}{dt}$$

exist and are continuous for every $t \geq 0, \varphi, \chi, \psi$.

Sufficient conditions that ensured stability, boundedness and asymptotic behavior of solutions of equations similar to the equation (1) or the system (2) were discussed in [1–3, 9, 17, 22]. However, the present work is a generalization and improvement on the works of Adams and Omeike [1], Ateş [9] and Omeike[17]. Particularly in [1], the case for which $w(t,\varphi,\chi,\psi)\chi = b(t)\phi(x,y)$ and $r(t) = 1$ in the system (2) was investigated for equiasymptotical stability of the zero solution.

Starting with the results published in ([1–3,22]), the aim of this paper is to obtain the afore-mentioned qualitative properties for equation (1) with much less restricted conditions and also generalize some earlier proved results in literature.

2. Main results

Henceforth, where necessary and for the sake of simplicity, we shall write

$$u(t,\varphi,\chi,\psi), \quad v(t,\varphi,\chi,\psi), \quad c(t,\varphi,\chi,\psi)$$

as $u(.)$, $v(.)$ and $c(.)$ respectively.

Theorem 1. *Assuming for some positive constants $\gamma_0, \gamma_1, \gamma_2, \alpha_c, \alpha_h, r_0, r_1$ the following conditions hold in addition to other basic conditions already assumed on non-linear terms appearing in* (1)*:*

(i) $\gamma_0 \leq w(t, \varphi, \chi, \psi) \leq \gamma_1, \quad \tau w_\varphi(t, \varphi, \tau, 0) \leq 0,$
$w_t(t, \varphi, \tau, 0) \leq 0, \quad w_\psi(t, \varphi, \chi, \psi)\chi \geq 0,$
for all $t \geq 0, \varphi, \chi, \psi$;

(ii) $0 < h(\varphi)\varphi^{-1} \leq \alpha_h$, $(\varphi \neq 0)$, $0 < c(.) \leq \alpha_c$;

(iii) $r_0 \leq r(t) \leq r_1$, $\dot{r}(t) \leq 0$;

(iv) $r(t) \leq \min\{\frac{2(v(.)+u(.))}{c(.)}, \gamma_0|\phi(t)|\}$ *and*
$1 + c(.)(h(\varphi)\varphi^{-1})^2 \leq |\phi(t)|$;

(v) $\Phi(t) \leq \gamma_2$;

(vi) $|p(t, \varphi, \chi, \psi)| \leq |\phi(t)|$

hold. Then system (2) *has all solutions to be uniformly ultimately bounded.*

Theorem 2. *Under the conditions of Theorem 1, every solution* $(\varphi(t), \chi(t), \psi(t))$ *of* (2) *is bounded uniformly and also satisfies*

$$\lim_{t\to\infty} \big(\varphi(t), \chi(t), \psi(t)\big) = (0, 0, 0).$$

Theorem 3. *If all the conditions of Theorem* (1) *hold, then any solution* $(\varphi(t), \chi(t), \psi(t))$ *of* (2) *passing through the initial condition*

$$\big(\varphi(0), \chi(0), \psi(0)\big) = (\varphi_0, \chi_0, \psi_0),$$

must satisfy

$$|\varphi(t)| \leq B, \quad |\chi(t)| \leq B, \quad |\psi(t)| \leq B, \tag{3}$$

for every $t \geq 0, \varphi, \chi$ *and* ψ*, where* $B > 0$ *is a constant.*

Theorem 4. *If* $p(t, \varphi, \chi, t) = 0$ *and conditions (i)-(v) listed in Theorem 1 hold. Then,* (2) *has uniformly asymptotically stable trivial solution.*

We prove Theorems 1 – 4 using the function $V(t, \varphi, \chi, \psi) = V(t)$ given by

$$V(t) = e^{-\Phi(t)} U_1(t, \varphi, \chi, \psi), \tag{4}$$

where,

$$\Phi(t) = \int_0^t |\phi(s)| ds,$$

and $U_1(t, \varphi, \chi, \psi) = U_1(t)$ is defined as

$$2U_1(t) = r(t)\varphi^2 + 2\int_0^\chi \tau w(t, \varphi, \tau, 0) d\tau + \psi^2 + 4. \tag{5}$$

First, we shall establish that the function contained in (4) is a Lyapunov function through the following lemma.

Lemma 1. *By Theorem (1), we can find certain positive constants B_0, B_1 and B_2 such that the scalar function $V(t)$ in (4) and it's derivative $\dot{V}(t)$ satisfy*

$$B_0\{X\} \leq V(t) \leq B_1\{X\},$$
$$V(t) \to +\infty \text{ as } X \to +\infty$$

and the derivative of $V(t)$ along (2) satisfies

$$\dot{V}_{(1.2)}(t) \leq -B_2\{X\},$$

where $X = \varphi^2(t) + \chi^2(t) + \psi^2(t)$.

Proof. We shall establish here that $U_1(t)$ in (5) is both bounded below and above by some positive functions.

$$\begin{aligned} 2U_1(t) &= r(t)\varphi^2 + 2\int_0^\chi \tau w(t,\varphi,\tau,0)d\tau + \psi^2 + 4 \\ &\geq r(t)\varphi^2 + 2\gamma_0\int_0^\chi \tau d\tau + \psi^2 \\ &\geq r_0\varphi^2 + \gamma_0\chi^2 + \psi^2 \\ &\geq B_0\{X\}, \end{aligned} \tag{6}$$

such that $B_0 = \min\{r_0, \gamma_0, 1\}$.

Similarly, by making use of the conditions of Lemma 1 or Theorem 1, we have

$$2U_1(t) \leq B_1\{X\} + 4, \tag{7}$$

where $B_1 = \max\{r_1, \gamma_1, 1\}$. Thus, by condition (v) of Theorem 1, inequalities (6) and (7), we have

$$B_0 e^{-\gamma_2}\{X\} \leq V(t) \leq B_1\{X\} + 4. \tag{8}$$

The first half of inequality (8) shows that $V(t)$ is positive definite for all φ, χ, ψ and

$$V(t) \to +\infty \text{ as } X \to \infty. \tag{9}$$

Next, we obtain $\dot{V}(t)$ along the system (2) with respect to variable t.

$$\begin{aligned} \dot{V}|_{(2)}(t) &= -\dot{\Phi}(t)e^{-\Phi(t)}U_1(t) + e^{-\Phi(t)}\dot{U}_1(t) \\ &= -e^{-\Phi(t)}\Big\{\dot{\Phi}(t)\Big(r(t)\varphi^2 + 2\int_0^\chi \tau w(t,\varphi,\tau,0)d\tau + \psi^2 + 4\Big) - \dot{r}(t)\varphi^2 \\ &\quad - 2r(t)\varphi\chi + 2(u(.) + v(.))\psi^2 + 2r(t)c(.)h(\varphi)\psi - 2\psi p(t,\varphi,\chi,\psi) \\ &\quad + [w(t,\varphi,\chi,\psi) - w(t,\varphi,\chi,0)]\chi\psi \\ &\quad - 2\chi\int_0^\chi \tau w_\varphi(t,\varphi,\tau,0)d\tau - 2\int_0^\chi \tau w_t(t,\varphi,\tau,0)d\tau\Big\} \\ &= -e^{-\Phi(t)}\Big\{|\phi(t)|\Big(r(t)\varphi^2 + 2\int_0^\chi \tau w(t,\varphi,\tau,0)d\tau + \psi^2 + 4\Big) - \dot{r}(t)\varphi^2 \end{aligned}$$

$$- 2r(t)\varphi\chi + 2(u(.) + v(.))\psi^2 + 2r(t)c(.)h(\varphi)\psi - 2\psi p(t,\varphi,\chi,\psi)$$
$$+ [w(t,\varphi,\chi,\psi) - w(t,\varphi,\chi,0)]\chi\psi - 2\chi\int_0^{\chi}\tau w_{\varphi}(t,\varphi,\tau,0)d\tau$$
$$- 2\int_0^{\chi}\tau w_t(t,\varphi,\tau,0)d\tau\Big\}.$$

By applying the conditions stated in Lemma (1) and noting that

$$0 \le (\psi - 2)^2,$$
$$2\psi \le \frac{1}{2}(\psi^2 + 4) \le \psi^2 + 4,$$

for all $t \ge 0, \psi$.

Hence,

$$2\psi|p(t,\varphi,\chi,\psi)| \le (\psi^2 + 4)|\phi(t)|. \tag{10}$$

Also, by using the inequality $|ab| \le \frac{1}{2}(a^2 + b^2)$, we have

$$\begin{aligned} |\phi(t)|r(t)\varphi^2 + 2r(t)|\varphi\chi| &\le |\phi(t)|r(t)\varphi^2 + r(t)\{\varphi^2 + \chi^2\}, \\ 2r(t)c(.)|h(\varphi)\psi| &\le r(t)c(.)\{h^2(\varphi) + \psi^2\}. \end{aligned} \tag{11}$$

Similarly, by mean value theorem and the condition (i) of Theorem (1), we get

$$\begin{aligned} w(t,\varphi,\chi,\psi)\chi\psi - w(t,\varphi,\chi,0)\chi\psi &= \frac{w(t,\varphi,\chi,\psi) - w(t,\varphi,\chi,0)}{\psi}\chi\psi^2 \\ &= w_{\psi}(t,\varphi,\chi,\theta\psi)\chi\psi^2 \\ &\ge 0, \end{aligned} \tag{12}$$

where $0 \le \theta \le 1$.

Therefore, from (10) – (12), we have

$$\dot{V}|_{(2)}(t) \le -e^{-\Phi(t)}\bigg\{\Big[r(t)\Big(|\phi(t)| - \big(1 + c(.)(h(\varphi)\varphi^{-1})^2\big)\Big) - \dot{r}(t)\Big]\varphi^2$$
$$+ \Big[|\phi(t)|\gamma_0 - r(t)\Big]\chi^2 + \Big[2(u(.) + v(.)) - r(t)c(.)\Big]\psi^2\bigg\}.$$

Hence, from conditions listed in (iv) of Theorem (1), we can get a positive constant say, B_3, so small that

$$\dot{V}_{(2)}(t) \le -B_3(X), \tag{13}$$
$$\forall \quad t \ge 0,\ x,\ y \text{ and } z,$$

where

$$X = x^2 + y^2 + z^2.$$ □

Proof of Theorem 1. Suppose that $(\varphi(t), \chi(t), \psi(t))$ is any solution to (2). Then, by estimates (8), (9) and (13), all the conditions Theorem 10.4, pp. 42 of Yoshizawa [28] hold. Hence, our result is established. □

Proof of Theorem 2. Suppose $(\varphi(t), \chi(t), \psi(t))$ is a given solution to the system (2). Then, by inequality (13) of Lemma 1, it is obvious that $\dot{V}_{(2)}(t) \leq 0$ for all $t \geq 0, \varphi, \chi, \psi$. Thus, $V(t)$ satisfied all the conditions of Theorem 10.2, pp. 38-39 of Yoshizawa [28]. Therefore, (2) has uniformly bounded solutions. The concluding part of the proof is established similarly as in the proof of Theorem (2) of [2]. Therefore, the details are omitted. $\square$

Proof of Theorem 3. Inequality (8) of Lemma 1 is still valid here while we have the derivative to be

$$\begin{aligned}\dot{V}|_{(2)}(t) = -e^{-\Phi(t)}\Big\{&|\phi(t)|\Big(r(t)\varphi^2 + 2\int_0^{\chi} \tau w(t,\varphi,\tau,0)d\tau + \psi^2 + 4\Big) - \dot{r}(t)\varphi^2\\ &- 2r(t)\varphi\chi + 2(u(.)+v(.))\psi^2 + 2r(t)c(.)h(\varphi)\psi - 2\psi p(t,\varphi,\chi,\psi)\\ &+ [w(t,\varphi,\chi,\psi) - w(t,\varphi,\chi,0)]\chi\psi\\ &- 2\chi\int_0^{\chi} \tau w_{\varphi}(t,\varphi,\tau,0)d\tau - 2\int_0^{\chi} \tau w_t(t,\varphi,\tau,0)d\tau\Big\}.\end{aligned}$$

After using (11) and (12), we obtain

$$\begin{aligned}\dot{V}|_{(2)}(t) \leq -e^{-\Phi(t)}\Big\{&\Big[r(t)\Big(|\phi(t)| - (1 + c(.)(h(\varphi)\varphi^{-1})^2)\Big) - \dot{r}(t)\Big]\varphi^2\\ &+ \Big[|\phi(t)|\gamma_0 - r(t)\Big]\chi^2 + \Big[2(u(.)+v(.)) - r(t)c(.) + |\phi(t)|\Big]\psi^2\\ &+ 4|\phi(t)| - 2\psi p(t,\varphi,\chi,\psi)\Big\}\\ \leq\ & e^{-\Phi(t)}2\psi p(t,\varphi,\chi,\psi).\end{aligned}$$

We make use of the fact that $2|\psi| \leq 1+\psi^2$ and the condition (vi) of Theorem (1), to obtain

$$\begin{aligned}\dot{V}|_{(2)}(t) &\leq e^{-\Phi(t)}\Big(1 + \psi^2(t)\Big)|p(t,\varphi,\chi,\psi)|\\ &\leq e^{-\Phi(t)}|\phi(t)| + e^{-\Phi(t)}\Big(\varphi^2(t) + \chi^2(t) + \psi^2(t)\Big)|\phi(t)|.\end{aligned}$$

By applying condition (v) of Theorem 1 and inequality (8) in the above estimate, we obtain

$$\dot{V}|_{(2)}(t) - B_0^{-1}|\phi(t)|V(t) \leq |\phi(t)|.$$

Solving this equation, we obtain

$$V(t) \leq B_4, \tag{14}$$

where $B_4 = [V(t_0) + \gamma_2]\exp(B_0^{-1}\gamma_2) > 0$ is a constant. Thus, from the left hand side of (8) and (14), we have

$$\{\varphi^2(t) + \chi^2(t) + \psi^2(t)\} \leq B_4 B_0^{-1}\exp\{\gamma_2\} = B_5. \tag{15}$$

Inequality (3) of Theorem 3 now follows from (15) with $B = \sqrt{B_5}$. $\square$

Proof of Theorem 4. Following the limit point approach of [27], we demonstrate that if Lemma 1 holds, then the function $U_1(t) \to 0$ as $t \to \infty$. First, we set $p = 0$ in (4) to obtain $V(t) = U_1(t)$. It follows immediately from (6) and (7) that $U_1(t)$ is positive definite and $U_1(t) \to \infty$ if and only if $\varphi^2(t)+\chi^2(t)+\psi^2(t) \to \infty$. The remaining part of the proof can be obtained just like in the proof of Theorem 2.1 of [4] or proof of Theorem 2.4 of [5]. $\square$

3. Example

An example which is a special case of either equation (1) or its equivalent system (2) will be provided to validate our results.

Consider the equation (1) and its equivalent system (2), if we let

$$w(t,\varphi,\chi,\psi) = 7 + \frac{2}{2+t^4+|\varphi\chi|\sin^2\chi + e^{-|\chi\psi|}},$$

$$v(t,\varphi,\chi,\psi) = 3 - \frac{2}{1+t^2+e^{-|\varphi\chi\psi|}},$$

$$u(t,\varphi,\chi,\psi) = 3 + \frac{1}{1+e^{t+\varphi^2+\chi^2+\psi^2}},$$

$$c(t,\varphi,\chi,\psi) = 1 + \frac{1}{2+t\varphi^2+\chi^2+\psi^2},$$

$$h(\varphi) = \frac{1}{4}\varphi, \quad r(t) = 4 + \frac{1}{4+t^4},$$

$$p(t,\varphi,\chi,\psi) = \frac{4}{2+\sin t + (\varphi+\chi+\psi)^2}.$$

We proceed to show that functions u, v, w, c, h and p satisfy all the assumptions placed on them under the basic assumptions.

Starting with function w, we observe that

$$7 \le w(t,\varphi,\chi,\psi) = 7 + \frac{2}{2+t^4+|\varphi\chi|\sin^2\chi + e^{-|\chi\psi|}} \le 8,$$

which gives $\gamma_0 = 7$ and $\gamma_1 = 8$.

The partial derives of w with respect to each of its dependent variables are:

$$w_t(t,\varphi,\chi,\psi) = -\frac{4t^3}{(2+t^4+|\varphi\chi|\sin^2\chi + e^{-|\chi\psi|})^2} \le 0,$$

$$w_\varphi(t,\varphi,\chi,\psi) = -\frac{|\chi|\sin^2\chi}{(2+t^4+|\varphi\chi|\sin^2\chi + e^{-|\chi\psi|})^2},$$

$$w_\psi(t,\varphi,\chi,\psi) = \frac{e^{-|\chi\psi|}|\chi|}{(2+t^4+|\varphi\chi|\sin^2\chi + e^{-|\chi\psi|})^2}.$$

Thus,

$$\tau w_\varphi(t,\varphi,\tau,0) = -\frac{\tau^2\sin^2\tau}{(2+t^4+|\varphi\tau|\sin^2\tau+1)^2} \le 0,$$
$$\chi w_\psi(t,\varphi,\chi,\psi) = \frac{e^{-|\chi\psi|}\chi^2}{(2+t^4+|\varphi\chi|\sin^2\chi+e^{-|\chi\psi|})^2} \ge 0.$$

Similarly, we have

$$c(t,\varphi,\chi,\psi) = 1+\frac{1}{2+t\varphi^2+\chi^2+\psi^2} \le 1.5 = \alpha_c,$$
$$3 \le u(t,\varphi,\chi,\psi) = 3+\frac{1}{1+e^{t+\varphi^2+\chi^2+\psi^2}},$$
$$1 \le v(t,\varphi,\chi,\psi) = 3-\frac{2}{1+t^2+e^{-|\varphi\chi\psi|}}.$$

More so, p satisfies

$$|p(t,\varphi,\chi,\psi)| \le \frac{4}{2+\sin t} = \phi(t),$$

which implies that

$$\frac{4}{3} \le |\phi(t)| \le 4.$$

Thus,

$$\Phi(t) = 4\int_0^t \frac{ds}{2+\sin s} = \frac{8\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{3}\big(2\tan(0.5t)+1\big)\right) - \frac{4\pi\sqrt{3}}{9}$$
$$\le 7.5 = \gamma_2.$$

$$h(\varphi)\varphi^{-1} \le \frac{1}{4} = \alpha_h.$$

Finally,

$$4 \le r(t) = 4+\frac{1}{4+t^4} \le \frac{17}{4},$$

such that, its derivative satisfies

$$\dot{r}(t) = -\frac{4t^3}{(4+t^4)^2} \le 0.$$

Clearly,

$$1+c(.)(h(\varphi)\varphi^{-1})^2 \le \frac{35}{32} \text{ and } \frac{4}{3} \le |\phi(t)|.$$

Hence,

$$1+c(.)(h(\varphi)\varphi^{-1})^2 \le |\phi(t)|.$$

$$r(t) \le \frac{17}{4} \le \min\left\{\frac{2(v(.)+u(.))}{c(.)}, \gamma_0|\phi(t)|\right\} = \min\left\{\frac{16}{3},\frac{28}{3}\right\} = \frac{16}{3}.$$

Therefore, all the conditions of the theorems are met by this example.

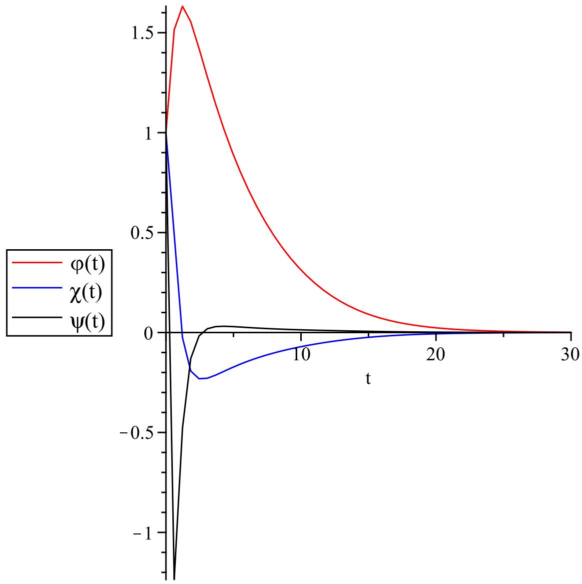

FIGURE 1. Showing uniform asymptotic stability behaviour of the zero solution when $p(t, \varphi, \chi, \psi) \equiv 0$ in the example constructed for $t \in [0, 30]$.

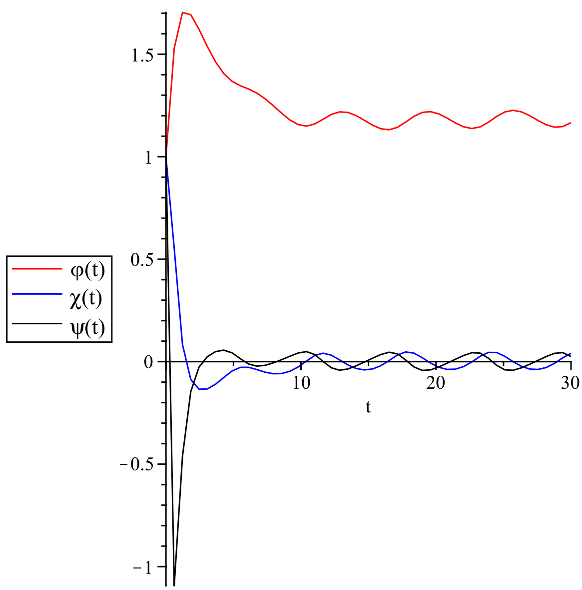

FIGURE 2. Showing uniform ultimate boundedness behaviour of all solutions when $p(t, \varphi, \chi, \psi) \neq 0$ in the example constructed for $t \in [0, 30]$.

4. Conclusion

By impossing conditions on the nonlinear terms appearing in the equation considered in this work, we have established sufficient criteria for the solutions of the problem to be stable and bounded using employing Lyapunov function as a tool.

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